

Centres of mass by integration

Recall from earlier, that

Fact — The centre of mass of a system of particles with positions \mathbf{x}_i and masses m_i satisfies:

$$\bar{\mathbf{x}} \sum m_i = \sum m_i \mathbf{x}_i$$

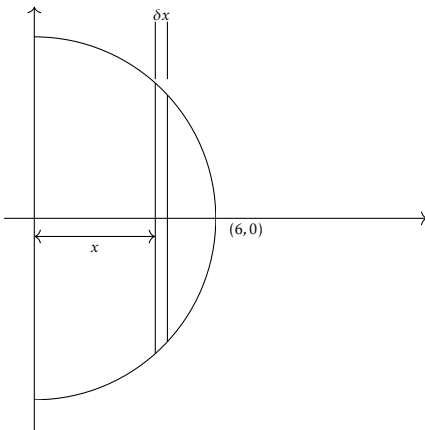
If we break down continuous shape into many small pieces, we these sums will tend to integrals, ie

Fact — The centre of mass of an object with *density* $m(\mathbf{x})$ at position \mathbf{x} , will have centre of mass $\bar{\mathbf{x}}$ satisfying:

$$\bar{\mathbf{x}} \int m(\mathbf{x}) d\mathbf{x} = \int m(\mathbf{x}) \mathbf{x} d\mathbf{x}$$

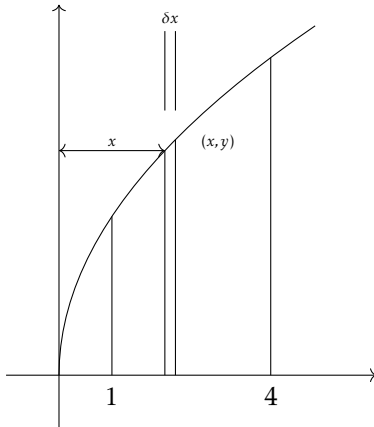
Example

A uniform semi-circular lamina has a radius of 6 cm and a mass per unit area m . Find the position of the center of mass.



Example

Find the centre of mass of a uniform lamina bounded by the curve $y^2 = 9x$, the x -axis and the coordinates $x = 1$ and $x = 4$, and lying in the first quadrant.



Fact — The coordinates of the centre of mass of a uniform lamina defined by the area between $y = f(x)$, $x = 0$, $x = a$ and the x -axis are:

$$\bar{x} \int_0^a f(x) dx = \int_0^a x f(x) dx \qquad \bar{y} \int_0^a f(x) dx = \frac{1}{2} \int_0^a f(x)^2 dx$$

Example

A lamina has three straight edges along the lines $x = 0$, $x = 2$, $y = 0$.

The fourth edge is a curve modelled by $f(x) = x^2 + 4$.

Find the coordinates of the centre of mass.

Example

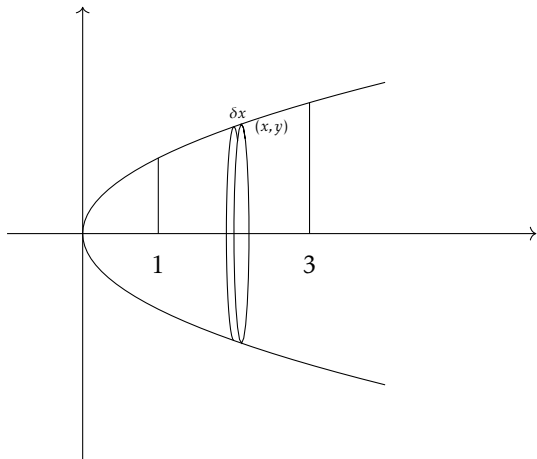
A *non-uniform* rod has length 8 m with mass density function $f(x) = 50 - x \text{ kgm}^{-1}$. Find the distance centre of mass from the denser end.

Solids of Revolution

By symmetry, the centre of mass must lie on the axis of rotation. The question then becomes, where do they lie on that axis?

Example

An area is enclosed by the curve $y^2 = 5x$, the x -axis, the lines $x = 1$, $x = 3$, and it lies in the first quadrant. The area is rotated about the x -axis through one revolution. Find the coordinates of the centre of mass of the uniform solid.

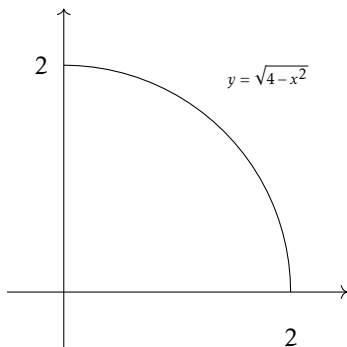


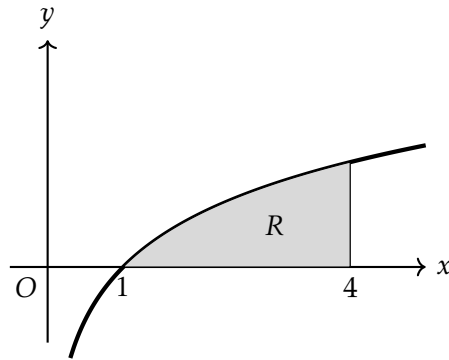
Fact — The centre of mass of a uniform solid of revolution with radius defined by $y = f(x)$ is:

$$\bar{x} = \frac{\int_0^a \pi x y^2 dx}{\int_0^a \pi y^2 dx}$$

Example

The region R is bounded by the curve $y = \sqrt{4-x^2}$ for $0 \leq x \leq 2$, the x -axis and the y -axis. R is rotated by 2π radians about the x -axis to produce a solid of revolution. Find the centre of mass.





Example (OCR M4 2017 Q4)

The diagram shows the curve with equation $y = \frac{1}{2} \ln x$. The region R , shaded in the diagram, is bounded by the curve, the x -axis and the line $x = 4$. A uniform solid of revolution is formed by rotation R completely about the y -axis to form a solid of volume V .

(i) Show that $V = \frac{1}{4}\pi(64\ln 2 - 15)$ [4]

(ii) Find the exact y -coordinate of the centre of mass of the solid [7]

First notice that $x = e^{2y}$

Rotating about the y axis we can consider the 'opposite' shape, which will have volume

$$\begin{aligned} V^c &= \pi \int_0^{\ln 2} x^2 dy \\ &= \pi \int_0^{\ln 2} e^{4y} dy \\ &= \pi \frac{1}{4} (16 - 1) = \frac{15}{4} \end{aligned}$$

The whole volume is $\pi \cdot 4^2 \cdot \ln 2 = 16\ln 2\pi$.

Therefore $V = 32\ln 2\pi - \frac{15}{4}\pi = \frac{1}{4}\pi(64\ln 2 - 15)$

To figure out the centre of mass, we can calculate strips, which will have width $4 - e^{2y}$ at height y , ie

$$\begin{aligned} V\bar{y} &= \int_0^{\ln 2} \pi(4^2 - e^{4y})y dy \\ &= \pi(8(\ln 2)^2 - \left[\frac{1}{4}ye^{4y}\right]_0^{\ln 2} + \int_0^{\ln 2} \frac{1}{4}e^{4y} dy) \\ &= \pi(8(\ln 2)^2 - 16\ln 2 + \frac{1}{16}(16 - 1)) \\ &= \pi(8(\ln 2)^2 - 16\ln 2 + \frac{15}{16}) \end{aligned}$$

$$\text{So } \bar{y} = \frac{8(\ln 2)^2 - 16\ln 2 + \frac{15}{16}}{\frac{1}{4}(64\ln 2 - 15)} = \frac{128(\ln 2)^2 - 256\ln 2 + 15}{256\ln 2 - 60}$$

Example

A body consists of a uniform solid circular cylinder C , together with a uniform solid hemisphere H which is attached to C . The plane face of H coincides with the upper plane face of C , as shown in the figure above. The cylinder C has base radius r , height h and mass $3M$. The mass of H is $2M$. The point O is the centre of the base of C .

(a) Show that the distance of the centre of mass of the body from O is

$$\frac{14h + 3r}{20}.$$

[5]

The body is placed with its plane face on a rough plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{4}{3}$. The plane is sufficiently rough to prevent slipping. Given that the body is on the point of toppling,

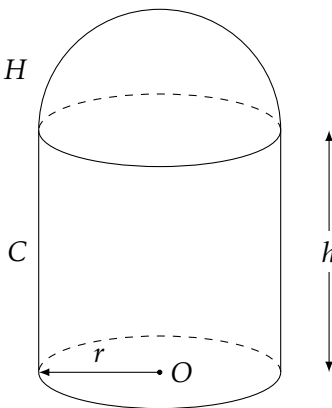
(b) find h in terms of r .

[4]

We know the centre of mass will lie on the axis of symmetry. We also have:

	mass	\bar{y}
cylinder	$3M$	$\frac{h}{2}$
hemisphere	$2M$	$h + \frac{3r}{8}$

Therefore



$$\begin{aligned} OG &= \frac{3Mh\frac{1}{2} + 2M(h + \frac{3r}{8})}{5M} \\ &= \frac{3h/2 + 2h + 3r/4}{5} \\ &= \frac{6h + 8h + 3r}{20} \\ &= \frac{14h + 3r}{20} \end{aligned}$$

We must have $\tan \alpha = \frac{r}{\frac{14h+3r}{20}}$, ie

$$\begin{aligned} \Rightarrow \quad \frac{4}{3} &= \frac{20r}{14h + 3r} \\ \Rightarrow \quad 14h + 3r &= 15r \\ \Rightarrow \quad 12r &= 14h \\ \Rightarrow \quad h &= \frac{6}{7}r \end{aligned}$$